情報論理 2012年 期末試験 9月3日

諸注意

- 全6問，問題は4ページある．
- 解答用紙に解答せよ．裏面等を使う場合は，その旨をはっきりわかるように記すこと．
- 答案には問題の番号を明記すること．
- 解答は日本語・英語のどちらで行ってもよい．英語のよく書けたものには加点を行う．
- ノート・参考書等の参照は不可．
- 不正行為には厳正に対処する．

Question 1.

(1) Below is the definition of *primitive recursive (PR)* functions. Fill in the blanks $\boxed{\text{(i)}}$–$\boxed{\text{(v)}}$.

(a) The function $\mathsf{zero} : \mathbb{N}^0 \longrightarrow \mathbb{N}$, given by $\mathsf{zero}() = 0$, is PR.

(b) The *successor* function $\mathsf{suc} : \mathbb{N} \longrightarrow \mathbb{N}$, given by $\mathsf{suc}(x) = \boxed{\text{(i)}}$, is PR.

(c) The *projection* function $\mathsf{proj}_i^n : \mathbb{N}^n \longrightarrow \mathbb{N}$, given by $\mathsf{proj}_i^n(x_1, \dots, x_n) = \boxed{\text{(ii)}}$, is PR. Here $i \in [1, n]$.

(d) Composition of PR functions is PR. That is,

$$\frac{g : \mathbb{N}^m \to \mathbb{N} \text{ is PR} \quad g_1 : \mathbb{N}^n \to \mathbb{N} \text{ is PR} \quad \cdots \quad g_m : \mathbb{N}^n \to \mathbb{N} \text{ is PR}}{g\big(g_1(\vec{x}), \dots, g_m(\vec{x})\big) \text{ is PR}}$$

(e) (Primitive recursion) Let $g : \mathbb{N}^n \to \mathbb{N}$ and $h : \mathbb{N}^{\boxed{\text{(iii)}}} \to \mathbb{N}$ be PR functions. Then the function $f : \mathbb{N}^{\boxed{\text{(iv)}}} \to \mathbb{N}$ defined by

$$f(\vec{x}, 0) := g(\vec{x}) \ , \qquad f(\vec{x}, y+1) := h(\vec{x}, y, \boxed{\text{(v)}})$$

is PR. Here $\boxed{\text{(iii)}}$ and $\boxed{\text{(iv)}}$ are the arities of h and f, respectively.

(2) Show that the *predecessor* function

$$\mathsf{pred}(x) := \begin{cases} x - 1 & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

is PR. (Hint: look carefully at how primitive recursion is formulated)

Question 2.

(1) Give an example of a recursive function that is not PR.

(2) Give an example of a recursively enumerable (RE) predicate that is not recursive.

(3) Answer if each of the following statements is true or false. (No need for proofs or counterexamples, though desired)

(a) If $P \subseteq \mathbb{N}^m$ is recursive, $\neg P$ (i.e. $\mathbb{N}^m \setminus P$) is recursive.

(b) If $P \subseteq \mathbb{N}^m$ is recursively enumerable, $\neg P$ (i.e. $\mathbb{N}^m \setminus P$) is recursively enumerable.

(4) What is "Negation Theorem"?

(5) If $P, Q \subseteq \mathbb{N}^m$ are recursively enumerable, $P \vee Q$ is recursively enumerable. Prove this.

(6) If $P, Q \subseteq \mathbb{N}^m$ are recursively enumerable, $P \wedge Q$ is recursively enumerable. Prove this.

Question 3.

Let $P \subseteq \mathbb{N}$ be a predicate; and consider the following conditions (a–c).

(a) There is a recursive function $g : \mathbb{N} \rightharpoonup \mathbb{N}$ such that

$$P = \mathrm{dom}(g) = \{x \in \mathbb{N} \mid \text{the value } g(x) \text{ is defined}\} .$$

(b) $P \subseteq \mathbb{N}$ is either empty, or there is a PR function $h : \mathbb{N} \to \mathbb{N}$ such that

$$P = \mathrm{image}(h) = \{h(x) \mid x \in \mathbb{N}\} .$$

(c) There exists a recursive predicate $Q \subseteq \mathbb{N}^2$ such that, for for any $x \in \mathbb{N}$,

$$P(x) \text{ holds} \iff Q(x, y) \text{ holds for some } y \in \mathbb{N} .$$

Answer the following questions.

(1) Show that (b) implies (c).

(2) Show that (c) implies (a).

(3) Show that (a) implies (b).

(Hint: transform g into the Kleene normal form. You would also need Gödel numbers of sequences.)

Question 4.

(1) Prove that, if

- $f : \mathbb{N}^m \rightharpoonup \mathbb{N}$ is a recursive function and
- its domain $\{\vec{x} \mid f(\vec{x}) \text{ is defined}\}$ is a recursive predicate,

then f can be extended into a *total* recursive function.

(2) Prove that, if $P \subseteq \mathbb{N}$ is RE and $g : \mathbb{N} \rightharpoonup \mathbb{N}$ is recursive, then the predicate

$$\{g(x) \mid x \in P\} \quad \subseteq \mathbb{N}$$

is RE.

Question 5.

(1) Explain each the following keywords using a few lines.

(a) Universal recursive function

(b) Recursion theorem

(2) Answer if each of the following statements is true or false. (No need for proofs or counterexamples, though desired)

(a) Given $k \in \mathbb{N}$, it is decidable if it is a code of a total recursive function.

(b) Given $k \in \mathbb{N}$, it is decidable if it is a code of some (not necessarily total) recursive function.

(c) Given natural numbers $k, l \in \mathbb{N}$, it is decidable if they are codes of the same recursive function.

Question 6.

Describe what you know about Gödel's *incompleteness* theorem. It can be:

- its intuition,
- its statement,
- a rough sketch of its proof,
- its relationship to Gödel's *completeness* theorem,

or else.