2012年度 情報論理 (0510014) 中間試験 6月18日

諸注意

- 全6問，問題は4ページある．
- 解答用紙に解答せよ．裏面等を使う場合は，その旨をはっきりわかるように記すこと．
- 答案には問題の番号を明記すること．
- 解答は日本語・英語のどちらで行ってもよい．英語のよく書けたものには加点を行う．
- ノート・参考書等の参照は不可．
- 不正行為には厳正に対処する．

Question 1.

Answer the following questions.

(1) Let (Σ, E) be the following algebraic specification.

$$\Sigma_0 = \{e\}\ ,\quad \Sigma_2 = \{\,\cdot\,\},\quad \Sigma_1 = \Sigma_3 = \Sigma_4 = \cdots = \emptyset\ ;$$
$$E = \{\mathbf{s}\cdot\mathbf{t} = \mathbf{t}\cdot\mathbf{s} \mid \mathbf{s},\mathbf{t} \text{ are } \Sigma\text{-terms}\} \cup \{e\cdot\mathbf{t} = \mathbf{t} \mid \mathbf{t} \text{ is a } \Sigma\text{-term}\}$$
$$\cup\, \{\mathbf{t}\cdot\mathbf{t} = \mathbf{t} \mid \mathbf{t} \text{ is a } \Sigma\text{-term}\}$$

Give an example of a (Σ, E)-algebra.

(2) Give explicit formulas for the following (capture-avoiding) substitutions.

$$\bigl(Q(x) \wedge \exists x.\, R(x,y)\bigr)[g(y)/x] \qquad \bigl(Q(y) \wedge \exists x.\, R(x,y)\bigr)[f(x)/y]$$

(3) Let $P, Q, R, \ldots$ be propositional variables. For each of the following propositional formulas, tell if it is valid or not. If it is valid, give a proof tree in LK; if it is not, give a valuation $J : \mathbf{PVar} \to \{\mathrm{tt}, \mathrm{ff}\}$ that makes the formula false. (You can consult Fig. 1 in the end)

(a) $(P \supset Q \supset R) \supset ((P \wedge Q) \supset R)$
(b) $((P \supset Q \supset P) \supset P$
(c) $((P \supset Q) \supset P) \supset P$

(4) For each of the following predicate formulas, tell if it is valid or not. If it is valid, give a proof tree in LK; if it is not, give a structure $\mathbb{S}$ and a valuation J over $\mathbb{S}$ that makes the formula false. Here P, Q, R are predicate symbols of the arity $0, 1, 2$, respectively. (You can consult Fig. 1 in the end)

(a) $\exists x.\forall y.\, R(x,y)\ \supset\ \forall y.\exists x.\, R(x,y)$
(b) $\forall x.\exists y.\, R(x,y)\ \supset\ \exists y.\forall x.\, R(x,y)$
(c) $\bigl(P \supset \forall x.\, Q(x)\bigr) \supset \exists x.\bigl(P \supset Q(x)\bigr)$

Question 2.

Let $\lesssim$ be a preorder (i.e. a reflexive and transitive relation) over a set X.

(1) Show that the relation $\lesssim \cap \gtrsim$ is symmetric.

The relation $\lesssim \cap \gtrsim$ is easily shown to be reflexive and transitive; hence it is an equivalence relation. We denote it by $\sim$.

(2) Show that we have

$$x \sim x' \;,\quad y \sim y' \;,\quad x \lesssim y \quad\Longrightarrow\quad x' \lesssim y' \;.$$

This means that the relation $\lesssim$ on the quotient set $X/\sim$, defined by

$$[x]_\sim \lesssim [y]_\sim \quad\overset{\text{def}}{\Longleftrightarrow}\quad x \lesssim y$$

is well-defined.

(3) Show that the induced relation $\lesssim$ on $X/\sim$ is antisymmetric.

The induced relation $\lesssim$ on $X/\sim$ is easily shown to be reflexive and transitive, too. Thus it is a partial order.

Question 3.

In propositional LK, we consider restricting the (Init) rule to the following one:

$$\frac{}{P \Rightarrow P}\ (\textsc{Init}),\ P \in \mathbf{PVar}$$

that is, only initial sequents with propositional variables are allowed. We denote the resulting deductive system by LK'. Show that LK and LK' has the same deductive power—that is, a formula is derivable in one system if and only if it is derivable in the other.

Question 4.

Let f be a 2-ary function symbol, and R be a 2-ary predicate symbol. Consider the following predicate formulas.

$$\begin{aligned}
A_1 &:\equiv \forall x.\, R(x,x) \\
A_2 &:\equiv \forall x,y.\, \bigl(R(x,y) \supset R(y,x)\bigr) \\
A_3 &:\equiv \forall x,y,z.\, \bigl(R(x,y) \supset R(y,z) \supset R(x,z)\bigr) \\
A_4 &:\equiv \forall x,y.\, \bigl(R(x,f(x,y)) \wedge R(y,f(x,y))\bigr) \\
A_5 &:\equiv \forall x,y,z.\, \bigl(R(x,z) \supset R(y,z) \supset R(f(x,y),z)\bigr)
\end{aligned}$$

(1) Present a structure $\mathbb{S}_1$ such that

$$\mathbb{S}_1 \models A_1 \wedge A_2 \wedge A_3 \wedge A_4 \wedge A_5 \;,$$

that is, a structure $\mathbb{S}_1$ that makes all the formulas $A_1, \ldots, A_5$ valid.

(2) Present a structure $\mathbb{S}_2$ such that

$$\mathbb{S}_2 \models A_1 \wedge A_2 \wedge A_3 \quad\text{but}\quad \mathbb{S}_2 \not\models A_4 \wedge A_5 \;.$$

Question 5.

In the first-order predicate logic, let us write $\mathrm{FV}(A)$ for the set of free variables in a formula A. Similarly we write $\mathrm{FV}(t)$ for the set of free variables in a term t.

(For simplicity, you can restrict the set of logical connectives to $\{\neg, \wedge, \forall\}$. You can forget about $\vee, \supset$, and $\exists$.)

(1) What are $\mathrm{FV}\big(f(g(x,y),z)\big)$ and $\mathrm{FV}\big(\forall y.\, P(x,y) \wedge Q(z)\big)$?

(2) Give a precise definition of $\mathrm{FV}(t)$ and $\mathrm{FV}(A)$ by induction.

(3) Prove the following two facts by induction.

(a) Let t be a term, $\mathbb{S}$ be a structure, and J, J' be valuations over $\mathbb{S}$ such that

$$J(x) = J'(x) \qquad \text{for each } x \in \mathrm{FV}(t).$$

Then we have $[\![t]\!]_{\mathbb{S},J} = [\![t]\!]_{\mathbb{S},J'}$.

(b) Let A be a formula, $\mathbb{S}$ be a structure, and J, J' be valuations over $\mathbb{S}$ such that

$$J(x) = J'(x) \qquad \text{for each } x \in \mathrm{FV}(A).$$

Then we have $[\![A]\!]_{\mathbb{S},J} = [\![A]\!]_{\mathbb{S},J'}$.

Question 6.

Describe what you know about "soundness" and "completeness." (The meaning of the words, their relationship, how one proves them, etc. Not too long; at most one page)

Initial sequents

$$\frac{}{A \Rightarrow A}\ (\textsc{Init})$$

Structural rules

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (\textsc{Weakening-L}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (\textsc{Weakening-R})$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (\textsc{Contraction-L}) \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ (\textsc{Contraction-R})$$

$$\frac{\Gamma, A, B, \Gamma' \Rightarrow \Delta}{\Gamma, B, A, \Gamma' \Rightarrow \Delta}\ (\textsc{Exchange-L}) \qquad \frac{\Gamma \Rightarrow \Delta, A, B, \Delta'}{\Gamma \Rightarrow \Delta, B, A, \Delta'}\ (\textsc{Exchange-R})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\textsc{Cut})$$

Logical rules

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{-L1})$$

$$\frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{-L2})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\text{-R})$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\vee\text{-L})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\text{-R1})$$

$$\frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\text{-R2})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Pi \Rightarrow \Sigma}{A \supset B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\supset\text{-L}) \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ (\supset\text{-R})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg\text{-L}) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ (\neg\text{-R})$$

$$\frac{A[t/x], \Gamma \Rightarrow \Delta}{\forall x.\, A, \Gamma \Rightarrow \Delta}\ (\forall\text{-L}) \qquad \frac{\Gamma \Rightarrow \Delta, A[z/x]}{\Gamma \Rightarrow \Delta, \forall x.A}\ (\forall\text{-R}),\ (\text{VC})$$

$$\frac{A[z/x], \Gamma \Rightarrow \Delta}{\exists x.\, A, \Gamma \Rightarrow \Delta}\ (\exists\text{-L}),\ (\text{VC}) \qquad \frac{\Gamma \Rightarrow \Delta, A[t/x]}{\Gamma \Rightarrow \Delta, \exists x.\, A}\ (\exists\text{-R})$$

Figure 1: Derivation rules of LK