100点
満点

2012年度 情報論理 (0510014) 中間試験 6月18日

諸注意

- 全6問，問題は4ページある．
- 解答用紙に解答せよ．裏面等を使う場合は，その旨をはっきりわかるように記すこと．
- 答案には問題の番号を明記すること．
- 解答は日本語・英語のどちらで行ってもよい．英語のよく書けたものには加点を行う．
- ノート・参考書等の参照は不可．
- 不正行為には厳正に対処する．

Question 1.

Answer the following questions.

5 (1) Let (Σ, E) be the following algebraic specification.

$$\Sigma_0 = \{e\}\ ,\quad \Sigma_2 = \{\,\cdot\,\},\quad \Sigma_1 = \Sigma_3 = \Sigma_4 = \cdots = \emptyset\ ;$$
$$E = \{\mathbf{s}\cdot\mathbf{t} = \mathbf{t}\cdot\mathbf{s} \mid \mathbf{s},\mathbf{t}\text{ are }\Sigma\text{-terms}\} \cup \{e\cdot\mathbf{t} = \mathbf{t} \mid \mathbf{t}\text{ is a }\Sigma\text{-term}\}$$
$$\cup\,\{\mathbf{t}\cdot\mathbf{t} = \mathbf{t} \mid \mathbf{t}\text{ is a }\Sigma\text{-term}\}$$

Give an example of a (Σ, E)-algebra. a 注

2+2 (2) Give explicit formulas for the following (capture-avoiding) substitutions.

$$\big(Q(x) \wedge \exists x.\, R(x,y)\big)[g(y)/x] \qquad \big(Q(y) \wedge \exists x.\, R(x,y)\big)[f(x)/y]$$

(3) Let $P, Q, R, \ldots$ be propositional variables. For each of the following propositional formulas, tell if it is valid or not. If it is valid, give a proof tree in LK; if it is not, give a valuation $J : \mathbf{PVar} \to \{\mathrm{tt}, \mathrm{ff}\}$ that makes the formula false. (You can consult Fig. 1 in the end)

typo (ごめん)

4 (a) $(P \supset Q \supset R) \supset ((P \wedge Q) \supset R)$ valid

4 (b) $\cancel{(}(P \supset Q \supset P) \supset P$ not valid (注 $P \supset Q \supset R$ は $P \supset (Q \supset R)$)

4 (c) $((P \supset Q) \supset P) \supset P$ valid

(4) For each of the following predicate formulas, tell if it is valid or not. If it is valid, give a proof tree in LK; if it is not, give a structure $\mathbb{S}$ and a valuation J over $\mathbb{S}$ that makes the formula false. Here P, Q, R are predicate symbols of the arity $0, 1, 2$, respectively. (You can consult Fig. 1 in the end)

4 (a) $\exists x.\forall y.\, R(x,y)\ \supset\ \forall y.\exists x.\, R(x,y)$ valid

4 (b) $\forall x.\exists y.\, R(x,y)\ \supset\ \exists y.\forall x.\, R(x,y)$ not valid

4 (c) $\big(P \supset \forall x.\, Q(x)\big) \supset \exists x.\big(P \supset Q(x)\big)$ valid

Question 2.

Let $\lesssim$ be a preorder (i.e. a reflexive and transitive relation) over a set X.

3 (1) Show that the relation $\lesssim \cap \gtrsim$ is symmetric.

The relation $\lesssim \cap \gtrsim$ is easily shown to be reflexive and transitive; hence it is an equivalence relation. We denote it by $\sim$.

3 (2) Show that we have

$$x \sim x' \ , \quad y \sim y' \ , \quad x \lesssim y \quad \Longrightarrow \quad x' \lesssim y' \ .$$

This means that the relation $\lesssim$ on the quotient set $X/\sim$, defined by

$$[x]_\sim \lesssim [y]_\sim \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad x \lesssim y$$

is well-defined.

4 (3) Show that the induced relation $\lesssim$ on $X/\sim$ is antisymmetric.

The induced relation $\lesssim$ on $X/\sim$ is easily shown to be reflexive and transitive, too. Thus it is a partial order.

Question 3.

In propositional LK, we consider restricting the (INIT) rule to the following one:

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$$\frac{}{P \Rightarrow P}\ (\textsc{Init}),\ P \in \mathbf{PVar}$$

that is, only initial sequents with propositional variables are allowed. We denote the resulting deductive system by LK'. Show that LK and LK' has the same deductive power—that is, a formula is derivable in one system if and only if it is derivable in the other.

Question 4.

Let f be a 2-ary function symbol, and R be a 2-ary predicate symbol. Consider the following predicate formulas.

$$\begin{array}{rl}
A_1 :\equiv & \forall x.\, R(x,x) \\
A_2 :\equiv & \forall x,y.\, \big(R(x,y) \supset R(y,x)\big) \\
A_3 :\equiv & \forall x,y,z.\, \big(R(x,y) \supset R(y,z) \supset R(x,z)\big) \\
A_4 :\equiv & \forall x,y.\, \big(R(x,f(x,y)) \wedge R(y,f(x,y))\big) \\
A_5 :\equiv & \forall x,y,z.\, \big(R(x,z) \supset R(y,z) \supset R(f(x,y),z)\big)
\end{array}$$

対称性

5 (1) Present a structure $\mathbb{S}_1$ such that

$$\mathbb{S}_1 \models A_1 \wedge A_2 \wedge A_3 \wedge A_4 \wedge A_5 \ ,$$

that is, a structure $\mathbb{S}_1$ that makes all the formulas $A_1, \ldots, A_5$ valid.

5 (2) Present a structure $\mathbb{S}_2$ such that

$$\mathbb{S}_2 \models A_1 \wedge A_2 \wedge A_3 \quad \text{but} \quad \mathbb{S}_2 \not\models A_4 \wedge A_5 \ .$$

Question 5.

In the first-order predicate logic, let us write $\mathrm{FV}(A)$ for the set of free variables in a formula A. Similarly we write $\mathrm{FV}(t)$ for the set of free variables in a term t.

(For simplicity, you can restrict the set of logical connectives to $\{\neg, \wedge, \forall\}$. You can forget about $\vee, \supset$, and $\exists$.)

4 (1) What are $\mathrm{FV}\big(f(g(x,y),z)\big)$ and $\mathrm{FV}\big(\forall y.\, P(x,y) \wedge Q(z)\big)$?

6 (2) Give a precise definition of $\mathrm{FV}(t)$ and $\mathrm{FV}(A)$ by induction.

(3) Prove the following two facts by induction.

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(a) Let t be a term, $\mathbb{S}$ be a structure, and J, J' be valuations over $\mathbb{S}$ such that
$$J(x) = J'(x) \qquad \text{for each } x \in \mathrm{FV}(t).$$
Then we have $[\![t]\!]_{\mathbb{S},J} = [\![t]\!]_{\mathbb{S},J'}$.

(b) Let A be a formula, $\mathbb{S}$ be a structure, and J, J' be valuations over $\mathbb{S}$ such that
$$J(x) = J'(x) \qquad \text{for each } x \in \mathrm{FV}(A).$$
Then we have $[\![A]\!]_{\mathbb{S},J} = [\![A]\!]_{\mathbb{S},J'}$.

Question 6.

12 Describe what you know about "soundness" and "completeness." (The meaning of the words, their relationship, how one proves them, etc. Not too long; at most one page)

$\boxed{\text{Initial sequents}}$

$$\frac{}{A \Rightarrow A}\ (\textsc{Init})$$

$\boxed{\text{Structural rules}}$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (\textsc{Weakening-L}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ (\textsc{Weakening-R})$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ (\textsc{Contraction-L}) \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ (\textsc{Contraction-R})$$

$$\frac{\Gamma, A, B, \Gamma' \Rightarrow \Delta}{\Gamma, B, A, \Gamma' \Rightarrow \Delta}\ (\textsc{Exchange-L}) \qquad \frac{\Gamma \Rightarrow \Delta, A, B, \Delta'}{\Gamma \Rightarrow \Delta, B, A, \Delta'}\ (\textsc{Exchange-R})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\textsc{Cut})$$

$\boxed{\text{Logical rules}}$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{-L1}) \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ (\wedge\text{-L2}) \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ (\wedge\text{-R})$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ (\vee\text{-L}) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\text{-R1}) \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}\ (\vee\text{-R2})$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Pi \Rightarrow \Sigma}{A \supset B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\supset\text{-L}) \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\ (\supset\text{-R})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg\text{-L}) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ (\neg\text{-R})$$

$$\frac{A[t/x], \Gamma \Rightarrow \Delta}{\forall x.\, A, \Gamma \Rightarrow \Delta}\ (\forall\text{-L}) \qquad \frac{\Gamma \Rightarrow \Delta, A[z/x]}{\Gamma \Rightarrow \Delta, \forall x.A}\ (\forall\text{-R}),\ (\text{VC})$$

$$\frac{A[z/x], \Gamma \Rightarrow \Delta}{\exists x.\, A, \Gamma \Rightarrow \Delta}\ (\exists\text{-L}),\ (\text{VC}) \qquad \frac{\Gamma \Rightarrow \Delta, A[t/x]}{\Gamma \Rightarrow \Delta, \exists x.\, A}\ (\exists\text{-R})$$

Figure 1: Derivation rules of LK

$\boxed{Q1}$

(1) - $X = \{0, 1\}$

例1 $[\![e]\!]_X = 1$

$[\![\cdot]\!]_X : X^2 \to X$, the usual multiplication of integers

- $X = \{0\}$

例2 $[\![e]\!]_X = 0$

$[\![\cdot]\!]_X : (0, 0) \mapsto 0$

(2)

$Q(g(y)) \wedge \exists x.\ R(x, \underset{\uparrow}{y})$

Do not replace a bound variable

$Q(f(x)) \wedge \underbrace{\exists z.\ R(z}_{\text{avoid capture}}, f(x))$

(3) (a) Valid.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow P} \qquad \dfrac{\dfrac{}{Q \Rightarrow Q} \quad \dfrac{}{R \Rightarrow R}}{Q \supset R,\ Q \Rightarrow R}(\supset\text{-}R)}{P \supset (Q \supset R),\ P,\ Q \Rightarrow R}(\supset\text{-}R)}{P \supset (Q \supset R),\ P \wedge Q,\ P \wedge Q \Rightarrow R}(\wedge\text{-}L)\ (\text{twice})}{P \supset (Q \supset R) \Rightarrow (P \wedge Q) \supset R}(\text{Contr.-}L)}{\Rightarrow (P \supset (Q \supset R)) \supset ((P \wedge Q) \supset R)}(\supset\text{-}R)
$$

(b) Not valid. Consider $\mathcal{J}$ s.t. $\mathcal{J}(P) = \mathrm{ff}$.

(c) Valid.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow P}}{P \Rightarrow P, Q}\,(\text{Weak.-R})}{\Rightarrow P,\ P \supset Q}\,(\supset\text{-R}) \qquad \dfrac{}{P \Rightarrow P}}{\dfrac{(P \supset Q) \supset P \Rightarrow P, P}{\dfrac{(P \supset Q) \supset P \Rightarrow P}{\Rightarrow ((P \supset Q) \supset P) \supset P}\,(\supset\text{-R})}\,(\text{Contr.-R})}\,(\supset\text{-L})$$

(4) (a) Valid.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{R(x,y) \Rightarrow R(x,y)}}{\forall y.\, R(x,y) \Rightarrow \exists x.\, R(x,y)}\,(\forall\text{-L})(\exists\text{-R})}{\exists x.\forall y.\, R(x,y) \Rightarrow \forall y.\exists x.\, R(x,y)}\,(\exists\text{-L})(\forall\text{-R})\ (\mathrm{VC})}{\Rightarrow \quad \ldots \quad (略)}\,(\supset\text{-R})$$

この順が逆
⇒ (VC) 不成立

(b) Not valid.

$U = \{0, 1\}$.

(diagonal)

$[\![R]\!] = \{(0,0), (1,1)\} = \Delta_U$

(c) Valid. (Pf tree は略)

$\boxed{\text{Q2}}$

(1) $(x, y) \in \lesssim \cap \gtrsim$

$\Leftrightarrow \quad x \lesssim y$ and $y \lesssim x$

$\Leftrightarrow \quad (y, x) \in \lesssim \cap \gtrsim$.

(2) By $x \sim x'$, $\quad x' \lesssim x$.
By $y \sim y'$, $\quad y \lesssim y'$.
By asmp., $\quad x \lesssim y$.

By transitivity of $\lesssim$, $\quad x' \lesssim y'$.

(3) Assume $[x]_\sim \lesssim [y]_\sim$ and $[y]_\sim \lesssim [x]_\sim$.

$x \lesssim y$, $y \lesssim x$.
Thus $x \sim y$, that is, $[x]_\sim = [y]_\sim$.

$\boxed{\text{Q3}}$

- LK' has less rules than LK, so

$$\vdash_{LK'} \Gamma \Rightarrow \Delta \quad \Longrightarrow \quad \vdash_{LK} \Gamma \Rightarrow \Delta$$

is obvious.

- For the opposite direction, it suffices to show that the (INIT) rule

$$\frac{}{A \Rightarrow A}\,(\text{INIT}) \quad \text{in LK} \quad \left(\begin{array}{c}A:\ \text{any}\\ \text{formula}\end{array}\right)$$

is admissible in LK'.

By induction on the construction of A.

$\boxed{Q4}$ (略)

$\boxed{Q5}$ 教科書の Lem. 4.3.6.
Lem. 4.3.7 の証明を参考に.

$\boxed{Q6}$ (略)